# Discrete Mathematics 2025 Spring

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CAMEA 中国高质量MBA教育认证 AACSB ACCREDITED EQUIS ACCREDITED

- 3.1 Basic Concepts of First-Order Logic
- 3.2 Equivalence Calculus of First-Order Logic

# 3.1 Basic Concepts of First-Order Logic



- 3.1.1 The Limitations of Propositional Logic
- 3.1.2 Individual Terms, Predicates, and Quantifiers
- 3.1.3 Symbolization of First-Order Logic
- 3.1.4 First-Order Logic Formulas and Classification

## ↳ The key limitations of propositional logic



- ***Inability to express individual relationships*** :

  It treats statements as whole units without using variables (***x***) and predicates (***H***(***x***), ***M***(***x***)) to represent properties.

  - For example, it can only express "**All humans are mortal**" as a single proposition, not as **∀x(*H*(*x*)→*M*(*x*))**, which specifies the relationship for each individual.

- ***Inability to perform generalization and instantiation reasoning*:** Propositional logic cannot express the general statement "All humans are mortal" (**∀x(*H*(*x*)→*M*(*x*))**, It cannot derive ***M***(***s***) from (**∀x(*H*(*x*)→*M*(*x*)** and ***H***(***s***), as it only combines concrete propositions without handling universal (∀) or existential (∃) quantification.

- **Lack of expressive power in predicate logic:**
  Propositional logic cannot describe relationships between different individuals (e.g., parent-child relationships, size comparisons).
  - For example, it cannot express the existential statement "Some people are mortal" $\exists x M(x)$.

e.g. **Example:** Theorem proof

**Suppose we have a mathematical theorem:**

If a number is even, then it is divisible by 2.

**Specific proposition:** "4 is even."

**Conclusion:** "4 is divisible by 2."

## ↳ The key limitations of propositional logic (cont.)



- **In first-order logic:**
  $\forall x(E(x)\rightarrow D(x))$ (All even numbers are divisible by 2).
  $E(4)$ (4 is even).
  **Conclusion:** D(4) (4 is divisible by 2).
- **Using propositional logic:**
  Let *P* represent "All even numbers are divisible by 2."
  Let *Q* represent "4 is even."
  Let *R* represent "4 is divisible by 2."
  The reasoning takes the form: $(P\wedge Q)\rightarrow R$.
- **Problem:**
  *P* is just a single proposition and cannot analyze its internal quantification $\forall x(E(x)\rightarrow D(x))$
  It cannot perform instantiation reasoning (e.g., it cannot directly derive "4 is divisible by 2" from "All even numbers are divisible by 2").

- 3.1.1 The Limitations of Propositional Logic
- 3.1.2 Individual Terms, Predicates, and Quantifiers
  - Individual Constants, Individual Variables, Domain of Discourse, Universe of Discourse
  - Propositional Constants, Propositional Variables
  - Universal Quantifiers, Existential Quantifiers
- 3.1.3 Symbolization of First-Order Logic
- 3.1.4 First-Order Logic Formulas and Classification

## ↳ Definition of an individual term



- ***Individual term***: Objects, either concrete or abstract, that can exist independently within the scope of study.
- ***Individual constant***: An individual term representing specific things, denoted by ***a***, ***b***, ***c***, etc.
- ***Individual variable***: An individual term representing abstract things, denoted by ***x***, ***y***, ***z***, etc.
- ***Domain of individuals***: The range of values that individual variables can take.
- ***Universal domain***: All things in the universe.
- **Such as**: "If ***x*** is even, then ***x*** is divisible by ***2***."
  - ***x***, even, and ***2*** are individual terms; even and ***2*** are individual constants; x is an individual variable.
  - The domain of individuals can be the set of natural numbers ***N***, the set of integers ***Z***, etc., or it can be the universal domain.

## ↳ Definition of a Predicate



- ***Predicate***: A word that represents the properties of individual terms or the relationships among them. Predicates are typically denoted by symbols such as ***F***, ***G***, ***H***, and ***P***.
- ***Predicate constant***: A predicate that represents a specific property or relationship.
- ***Predicate variable***: A predicate that represents an abstract property or relationship.
- ***n-ary predicate*** $P(x_1, x_2, \ldots, x_n)$: A predicate with ***n*** propositional variables, defined as an ***n*-ary** function over the domain of individuals with a range of {0, 1}.
- ***Unary predicate***: Represents properties of things.
- ***Multi-ary predicate*** (***n*** ≥ 2): Represents relationships among things.
- ***0-ary predicate***: A predicate without individual variables, i.e., a propositional constant or propositional variable.

**(1) 4 is even**

① "4" is an individual constant.② "is even" is a predicate constant.③ "4 is even " is symbolized as ***F*(4).**

**(2) Xiao Wang and Xiao Li are the same age**

① "Xiao Wang" and "Xiao Li" are individual constants. ② "are the same age" is a predicate constant. ③ Let ***a***: Xiao Wang, ***b***: Xiao Li, and ***G*(*x*,*y*)**: ***x*** and ***y*** are the same age.④ Symbolization as ***G*(*a*,*b*)**.

**(3) $x<y$**

① ***x*** and ***y*** are propositional variables."**<**" is a predicate constant. ② Symbolization as ***L*(*x*,*y*)**.

**(4) *x* has a certain property *P***

① ***x*** is a propositional variable. ② **P** is a predicate variable. ③ Symbolization as ***P*(*x*)**.

e.g. Example: Symbolize the following propositions using **0th-order predicates** and discuss their truth values.

**(1) $\sqrt{2}$ is irrational, and $\sqrt{3}$ is rational**

**(2) If 2>3, then 3<4**

- **Solve:**

**(1) Let *F*(*x*): *x*** is irrational, ***G*(*x*): *x*** is rational

Symbolization as: $F(\sqrt{2}) \wedge G(\sqrt{3})$, **True value: 0**

**(2) Let *F*(*x*,*y*): *x*>*y*, *G*(*x*,*y*): *x*<*y*,**

Symbolization as: $F(2,3) \rightarrow G(3,4)$, **True value: 1**

- ***Quantifiers***: Words that indicate quantity.
- ***Universal quantifier*** $\forall$: Indicates **'any'**, **'all'**, **'every'**, etc.

  **Such as:** $\forall x$ means "for all $x$ in the domain of discourse."

  $\forall xF(x)$ means "All $x$ have the property $F$."
- **Existential quantifier** $\exists$: Indicates **'there exists'**, **'there is'**, **'at least one'**, etc.

  **Such as:**

  $\exists x$ means "there exists an $x$ in the domain of discourse."

  $\exists xF(x)$ means "There exists an $x$ that has the property $F$.

## ↳ Property Predicate vs. Unary Predicate



- A ***property predicate*** can be regarded as a subset or a special case of a ***unary predicate***, i.e., ***Property Predicate* ⊆ *Unary Predicate***.

| Item | Property Predicate $P(x)$ | Unary Predicate $P(x)$ |
|---|---|---|
| **Definition** | Describes an **individual's** inherent property or characteristic | Any predicate that applies to a **single variable** |
| **Relation** | Describes **only the property of an object**, without involving other objects | May describe an object's property **or its role/relationship** |
| **Examples** | Even(x) (x is even)<br>Red(x) (x is red) | Student(x) (x is a student)<br>Parent(x) (x is a parent) |
| **Application** | used to express **individual properties** | A general term for **any predicate with one parameter** |

In machine learning model design, **property predicates** are typically used for classification tasks and can be utilized in **feature selection**, while **unary predicates** may involve object roles and **relational inference**.

## ↳ Property Predicate vs. First-order Logic(e.g.)



e.g. **Examples:** Symbolizing the following propositions in first-order logic:

(1) All people love beauty; (2) There is someone who writes with their left hand. **Domain of discourse:** (a) The set of humans; (b) The universal domain.

- **Solution: Domain of** (a) :

(1) Let $F(x)$: $x$ loves beauty. Symbolized as $\forall x\, F(x)$.

(2) Let $G(x)$: $x$ writes with their left hand. Symbolized as $\exists x\, G(x)$.

- **Solution: Domain of** (b) : Let $M(x)$: $x$ is a person. $F(x)$ and $G(x)$ are the same as in (a).

  (1) Symbolized as $\forall x\, (M(x) \rightarrow F(x))$.

  (2) Symbolized as $\exists x\, (M(x) \wedge G(x))$.

- $M(x)$ is called a **characteristic predicate.**

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e.g. **Examples:** Symbolize the following propositions and discuss their truth values:

(1) For all $x$, $x^2-3x+2=(x-1)(x-2)$; (2) There exists an x such that $x+5=3$.

Consider the following **domains** of discourse: (a) $D_1$=N (the set of natural numbers); (b) $D_2=R$ (the set of real numbers).

- **Solution:** Let $F(x)$: $x^2-3x+2=(x-1)(x-2)$, and $G(x)$: $x+5=3$.
  - (a) Domain $D_1$=N :
    - (1) $\forall x F(x)$ Truth value: 1 (True)
    - (2) $\exists x G(x)$ Truth value: 0 (False)
  - (b) Domain $D_2=R$ :
    - (1) $\forall x F(x)$ Truth value: 1 (True)
    - (2) $\exists x G(x)$ Truth value: 1 (True)

e.g. **Example:** Symbolize the following propositions:

(1) Rabbits run faster than turtles.

(2) Some rabbits run faster than all turtles.

(3) Not all rabbits run faster than turtles.

(4) There do not exist rabbits and turtles that run at the same speed.

- **Solution:** Using a universal domain, let:
  - $F(x)$: $x$ is a rabbit.
  - $G(y)$: $y$ is a turtle.
  - $H(x,y)$: $x$ runs faster than $y$.
  - $L(x,y)$: $x$ and $y$ run at the same speed.

(1) $\forall x \forall y(F(x) \wedge G(y) \rightarrow H(x,y))$

(2) $\exists x(F(x) \wedge \forall y(G(y) \rightarrow H(x,y)))$

(3) $\neg \forall x \forall y(F(x) \wedge G(y) \rightarrow H(x,y))$

(4) $\neg \exists x \exists y(F(x) \wedge G(y) \wedge L(x,y))$

- The use of **unary** and ***n*-ary** predicates.
- The **distinction** between universal and **existential** quantifiers.
- When multiple quantifiers appear, their **order** cannot be arbitrarily swapped.

  **Such as:** in the domain of real numbers $R$, **let** $H(x,y)$: $x+y=10$

  - $\forall x \exists y H(x,y)$ has a truth value of **1**
  - $\exists y \forall x H(x,y)$ has a truth value of **0**
- The symbolization of propositions is **not unique**.

**Such as:** based on the previous example, there are the following alternative symbolizations:

(1) $\forall x(F(x)\rightarrow\forall y(G(y)\rightarrow H(x,y)))$ $(\forall x\forall y(F(x)\wedge G(y)\rightarrow H(x,y)))$

(3) $\exists x\exists y(F(x)\wedge G(y)\wedge\neg H(x,y))$ $(\neg\forall x\forall y(F(x)\wedge G(y)\rightarrow H(x,y)))$

(4) $\forall x\forall y(F(x)\wedge G(y)\rightarrow\neg L(x,y))$ $(\neg\exists x\exists y(F(x)\wedge G(y)\wedge L(x,y)))$

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  - First-order language (alphabet, terms, atomic formulas, compound formulas)
  - Scope and bound variables, bound occurrences and free occurrences
  - Closed formulas
  - Interpretation of first-order language
  - Tautologies, contradictions, and satisfiable formulas
  - Substitution examples